NOISE MODELING AND RELIABILITY OF BEHAVIOR
PREDICTION FOR MULTI-STABLE HYDROELASTIC SYSTEMS

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ABSTRACT

This paper reviews results of experiments conducted on a simple multi-stable hydroelastic (galloping) oscillator. These results show that noise may cause a multi-stable hydroelastic system to exhibit chaotic behavior, and that in some instances such behavior cannot be predicted reliably unless noise effects are carefully accounted for. We then present results of a theoretical investigation of a simple, paradigmatic multi-stable system, the Duffing-Holmes oscillator. The results of this investigation show that for the system being considered noise promotes the occurrence of chaotic behavior associated with Smale horseshoes. This theoretical investigation is the first phase of an effort to develop analytical tools for predicting reliably the potential for chaotic behavior of actual hydroelastic systems such as deep-water compliant platforms.

INTRODUCTION

Overlooking certain types of dynamic behavior can constitute a gross design error with possibly disastrous consequences. It is such a design error that led in 1836 to the collapse by flutter of the Brighton Chain Pier bridge. Neither this precedent nor the development of flutter theory by Theodorsen (1935) were understood by the bridge design community, and a similar design error led to the well-known collapse in 1940 of the Tacoma-Narrows bridge.

Thompson et al. (1984) noted that experiments and numerical studies conducted on compliant offshore structures can reveal the possible occurrence of unexpected types of dynamic behavior, including deterministic chaos. In those studies the chaotic behavior of the system of interest occurred in a hydroelastic system excited by periodic loads. More recently, experiments and numerical studies performed on a simple, paradigmatic hydroelastic system showed that irregular behavior involving catastrophic jumps between distinct regions of phase space could also be induced by the noise excitation of a multi-stable system (Simiu and Cook, 1991, 1992). It is this finding that motivated the present work.

Indeed, new types of compliant offshore structures are being envisaged that can be anticipated to exhibit increasingly complex nonlinear behavior, including multi-stability. The hydroelastic studies just alluded to are therefore likely to be relevant in this context. This suggests the need to develop tools for describing and understanding dynamic effects due to the presence of noise. Numerical simulations can be helpful for this purpose. However, as pointed out in the paper, the numerical simulation approach has a number of practical limitations. It is reasonable to expect that theory could overcome some of these limitations, offer valuable fundamental insights, and help to develop reliable prediction capabilities.

Mathematically, a distinction has been made in the literature between chaotic (i.e., irregular) "basin-hopping" behavior with jumps induced by stochastic excitation or noise (Arecchi et al., 1983), and behavior exhibiting jumps associated with deterministic chaos (i.e., behavior that entails the formation of Smale horseshoes and, therefore, sensitivity to initial conditions, at least one positive Lyapounov exponent, basins of attraction with fractal dimension, and the existence of a strange attractor with fractal dimension (Guckenheimer and Holmes, 1983)). Experimental and numerical results (Simiu and Cook, 1992) have shown, however, that these two types of behavior can be indistinguishable phenomenologically.

In this paper we present a theoretical investigation which shows that, for a certain class of systems and for certain regions of parameter space, the mathematical distinction between noise-induced (stochastic) chaos and deterministic chaos can in fact be artificial. That is, what appears to be basin hopping caused by any given realization of a noisy process can in fact be a form of chaotic behavior associated with the formation of Smale horseshoes. Basic theory yields in this case necessary conditions for the occurrence of such chaotic behavior, as well as a useful measure of its strength as reflected,
for example, in the frequency of the jumps.

We emphasize, however, that at this point our theoretical results are not directly applicable to hydroelastic structures. In spite of the useful insights they provide, they should be viewed as representing a first step toward developments applicable to wider classes of systems, including hydroelastic systems.

Following a brief review of relevant experimental and numerical results, we focus our attention on a class of systems for which available theory allows the development of useful analytical tools. We then present results yielded by those tools, which include the generalized Melnikov function and the phase space flux. We then comment on the usefulness and limitations of the approaches discussed in the paper.

NOISE-INDUCED JUMPS IN A HYDROELASTIC SYSTEM

In this section we review briefly experimental and numerical results obtained in the study of a galloping system. The system consisted of two elastically restrained and elastically coupled horizontal square bars immersed in a uniform horizontal water flow. The upstream faces of the two bars were contained in a vertical plane normal to the flow velocity. Drag wires constrained the bars to oscillate in an arc with large radius tangent to the vertical plane (Fig. 1) — see Simiu and Cook, (1991, 1992) for details.

Any deviation of a bar from its position of equilibrium causes self-excited lift (galloping) forces that result in an increase of that deviation, that is, the position of equilibrium is an unstable fixed point, so that for small deviations the hydrodynamic damping inherent in the lift is negative. For larger deviations the hydrodynamic damping becomes positive. This limits the amplitude of the oscillation, which thus describes a limit cycle.

In addition to the galloping forces the bars are subjected to forces induced by vortices shed in their wakes. Observations showed that, for relatively low reduced flow velocities, the two bars oscillate in phase, that is, both bars move together up or down. As the reduced velocity grows, in-phase oscillations alternate irregularly with opposite-phase oscillations, where one of the bars moves up while the other moves down. An example is shown in Fig. 2, which depicts displacements (in meters) of the top prism, and Fig. 3, which depicts displacements of both the top and bottom prism for a 5 s interval of Fig. 2.

Motions such as those of Figs. 2 and 3 are reminiscent of the irregular alternation between different oscillatory forms of a forced magnetoelastic beam (Moon and Holmes, 1979) or a forced buckled column (Cook and Simiu, 1991). The motions are irregular, and for any given system the frequency of the alternations increases with the flow velocity. A visual examination of the time histories is not sufficient to reveal whether the irregularity of the motion is due to the randomness of the excitations or to chaoticity associated with Smale horseshoes. For convenience we refer here to the alternations between in-phase and out-of-phase oscillations depicted in Figs. 2 and 3 as irregular jumps.

A dependable numerical simulation of the galloping motions of this system would in principle require the
solution of the Navier-Stokes equations. Because such a solution would entail considerable if not inaperable difficulties, engineers resort instead to semi-empirical models for the hydrodynamic loads. Thus, following Parkinson and Smith (1964), the self-excited lift forces are described by nonlinear functions of the angle of attack, with Reynolds-number dependent constant coefficients obtained from measurements under static conditions. This model is clearly imperfect, though for certain applications it yields acceptable results. The vortex-induced lift forces are also described empirically, a commonly used description being given by the so-called lift oscillator model — see, e.g., Simiu and Scanlan (1986, p. 202). In addition, random forces are acting on the bars. These are associated with flow separation (both spanwise and at the bars' ends), oncoming and wake flow turbulence, and irregular flow around ancillary parts. The modeling issue for these random forces is difficult and, in many cases, it has not been resolved satisfactorily to date.

Attempts to reproduce observed motions with irregular jumps by omitting random forces from the hydrodynamic loading model were reported in some detail by Simiu and Cook (1992). (With the random forces absent, the oscillator is modeled by an eighth-order autonomous differential system.) In some isolated cases those attempts were successful, that is, chaotic behavior was obtained by solving the differential system numerically. However, most simulations failed to predict reliably the occurrence of jumps. Simulations were then attempted in which random forces acting on the bars were included. For simplicity this was done by assuming those forces to be modeled by white noise, with intensity chosen by trial and error to yield time histories comparable to those observed in the laboratory. Thus, whereas a hydrodynamic model from which random fluctuations were omitted had neither predictive nor descriptive capabilities, a model in which a crude representation of such fluctuations was included was capable of describing — though not predicting — the observed motions fairly adequately (Simiu and Cook, 1991).

The results thus show, at least for certain types of multi-stable systems, that to predict motions with irregular jumps it is necessary to model the actual hydrodynamic forces, including their random parts. Once this is done, simulations can help to assess the tendency of the system to experience catastrophic jumps induced by random excitation (or noise).

### EFFECT OF NOISE ON A MULTI-STABLE SYSTEM: THEORY

Motivated by the results reviewed in the preceding section, we sought to develop a theoretical approach to the prediction of irregular jumps in a nonlinear system. No general theoretical approach to this problem exists. However, for dynamical systems with a global geometry of a type described below, Melnikov theory and the related notion of phase space flux do provide useful theoretical insights. Restricting ourselves to dynamical systems of that type, we study the effects of noise on the system's sensitivity to chaos. This is the first phase in a planned effort to develop analytical tools applicable to structures of interest in ocean engineering.

We now describe our model and briefly introduce some of the fundamentals of Melnikov theory and phase space flux.

#### Dynamical Model

The systems we consider consist of second-order perturbed differential equations whose unperturbed flows include homoclinic or heteroclinic orbits. For definiteness we focus our attention on a typical example of such a system, the Duffing-Holmes oscillator. Its equation is

\[ \ddot{x} - x - x^3 = \varepsilon [\cos(\omega t) + sG_t - k x] \]  

(1)

where \( \varepsilon \) is a small number, \( \gamma, \sigma, k \) and \( \omega \) are constants, and

\[ G_t = \left[2/N\right]^{1/2} \sum_{n=1}^{N} \cos(\omega_n t + \phi_n) \]  

(2)

where \( \omega_n, \phi_n : n=1,2,\ldots,N \) are independent random variables, \( \omega_n : n=1,2,\ldots,N \) are positive and identically distributed with density \( \Psi \), and \( \phi_n : n=1,2,\ldots,N \) are identically uniformly distributed over the interval \([0,2\pi]\). \( N \) is a fixed parameter of the model assumed to be finite, though it may be arbitrarily large.

#### Shinozuka Noise.

The process \( G \) defines Shinozuka noise (Shinozuka, 1971). It is uniformly bounded with zero mean and unit variance. For large \( N \), \( G \) is nearly Gaussian and has one-sided spectral density \( 2\pi \Psi \). Thus, by properly choosing the density of \( \omega_n \), Shinozuka noise can have any specified spectral density. We choose Shinozuka noise over other types of noise commonly employed in engineering applications, for example Nyquist noise (Rice, 1954), because it is uniformly bounded. This property is essential for the application of Melnikov theory.

#### Melnikov function.

According to Melnikov theory, a necessary condition for the occurrence of chaos in a system such as Eq. 1 is that its generalized Melnikov function have simple zeros (Wiggins, 1988, p. 463). (Arrowsmith and Place, 1990, p. 174).

For Eq. 1 the generalized Melnikov function has the expression (Wiggins, 1988, p. 463):

\[ M(t_1, t_2) = -3k/4 + S(\omega) \sin(\omega t_1 + \phi_1) + Z_{a_2} \]  

(3)

with

\[ S(\omega) = (2)^{1/2}\text{ cosech}(\omega \Delta/2) \]  

(4)

and

\[ Z_{a_2} = a N^{1/2} \sum_{n=1}^{N} S(\omega_n) \sin(\omega_n t_2 + \phi_n) \]  

(5)

From the general form of the expression for the Melnikov function (Arrowsmith and Place, 1990) it follows immediately that \( Z_{a_2} \) is the result of the convolution of \( G(t_2) \) with \( x_n(-t) \), where

\( (x_n(t), \dot{x}_n(t)) = (2)^{1/2}\text{ sechh}, -(2)^{1/2}\text{ sechh tanh}) \)

are the phase plane coordinates of the homoclinic orbit of the unperturbed Duffing-Holmes oscillator (Wiggins, 1990, p. 513). Consequently \( x_n(-t) \) may be interpreted as an impulse response function. (A similar observation can be made for the second term in the r.h.s. of Eq. 3.) Since the expectation of \( G(t_2) \) is zero, so is the expectation of \( Z_{a_2} \). The variance of \( M(t_1, t_2) \) is easily obtained by noting that the spectral density of \( Z_{a_2} \) is equal to \( S^2(\omega) \) times the spectral density \( \Psi(\omega) \), since \( S^2(\omega) \) may be interpreted as the square of the modulus of the transfer function corresponding to the impulse response function \( x_n(-t) \). Thus,

\[ \sigma^2 = \sigma^2 \int_{-\infty}^{\infty} S^2(\omega) \Psi(\omega) d\omega \]  

(6)
The flux increases with the excitation amplitude \( \gamma \), as expected. Figure 5 shows time histories of the displacement \( x \) for four values of \( \gamma \) to which there correspond chaotic motions with snapthrough (jumps). (Figures 4 and 5 were obtained by a computer program described by Parker and Chua (1989)).

\[ \Phi = \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} M'(s, \theta_1, \theta_2) ds \]  

\[ \Phi = E[(\sigma^2 A + 1/2 - 1)\eta] \]  

Table 1 shows, for each value of \( \gamma \), the estimated mean escape time (time between successive jumps) in nondimensional units (based on a time interval \( T = 20,000 \)), and the ratio \( \Phi(\gamma)/\Phi(\gamma=0.400) \), which was estimated numerically.

<table>
<thead>
<tr>
<th>( \gamma )</th>
<th>Mean Escape Time</th>
<th>Ratio ( \Phi(\gamma)/\Phi(\gamma=0.400) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.400</td>
<td>120</td>
<td>1.000</td>
</tr>
<tr>
<td>0.425</td>
<td>55</td>
<td>1.062</td>
</tr>
<tr>
<td>0.450</td>
<td>35</td>
<td>1.126</td>
</tr>
<tr>
<td>0.475</td>
<td>28</td>
<td>1.187</td>
</tr>
</tbody>
</table>
At this time numerical simulations are the only available
harmonic or quasiperiodic function, (2) a realization of
whether noise raises or lowers the threshold for the
basins of attraction and the attractors in phase space.
Numerical Approach
occurrence of chaos in a system whose unperturbed
proximity of an attractor to a separatrix increases
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behavior of the Melnikov function, and the magnitude of
jumps) is related to the flux factor function. Those
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excitation can play an important role in determining the behavior of the system.
For the class of systems considered, the necessary
criterion for the occurrence of chaos is related to the
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