Vorticity Transport Integral Concept for Determining Wave Forces on Submerged Bodies

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The vorticity transport integral (VTI) concept affords an analytical procedure that is well suited to determining the wave forces on bodies and structural elements in a marine environment. The concept can be applied to any unsteady flow; however, it is especially suited to the treatment of flows associated with ocean waves in which an object experiences complete flow reversal with wake motion confined to the vicinity of the body. As an integral approach it allows forces to be determined accurately with either an approximate description or an exact solution of the flow. Because the integrals have significant values only near the body, the approach avoids the problems inherent with the evaluation of integrals over large volumes of fluid. The resulting expressions are used in a qualitative appraisal of the various contributions to the total force on a body in an unsteady flowfield.

Introduction

Very simply stated, the question under study is: How can forces due to the ocean wave interaction with marine structures be estimated to a sufficient degree of certainty? The ocean waves in question can range from well-behaved swells to a complex sea state due to the interaction of currents and large amplitude waves arriving from different directions. The marine structures of interest may vary considerably in size, geometry, and degree of immersion. Structures of particular interest in marine applications are the relatively stationary components of offshore platforms, floating vessels, underwater facilities, marine risers, pipelines, etc.

Our study is based upon the premise that wave forces on structures can be predicted accurately providing descriptions of the far-field kinematics (the ocean wave or sea state) and the near-field kinematics (the fluid-structure interaction) are available. The description ultimately must account for the viscous flow pattern considering flow separation while also accounting for wake sweeping, inclination of the structural elements, and variations of the flow along the length of the structural elements. It is not within the scope of the present study to remark on the various models that have been proposed to account for these near-field phenomena, but, rather, our purpose has been to develop a concept which can be used to 1) explore qualitatively the influences of the various near-field phenomena that have been argued to be significant, 2) compare quantitatively the importance of the phenomena and the model used to describe it, and 3) develop a pragmatic methodology with which forces can be estimated.

Our objective has been to introduce and develop a new concept for determining the hydrodynamic forces on submerged bodies in an unsteady far-field flow. The Vorticity Transport Integral (VTI) concept is an indirect analytical approach which yields a rational basis for calculating forces on the surfaces of a structure due to an unsteady flow. The approach is an indirect approach (as is a control volume approach) in which the forces are expressed as integrals over the unsteady flowfield. Because the resulting integrals are expressed in terms that represent defects in the far-field flow (similar to the momentum integral method of boundary-layer theory) their evaluation is tractable. The approach is capable of taking into account as many details of a transient flow as an investigator is willing to describe. For example, wake sweeping, asymmetry in the flow separation pattern, and other three-dimensional aspects of a time-dependent viscous flow can be considered ultimately.

Although the presentation of the concept for a general flow and system orientation is our goal, this initial exposition is restricted to a time-dependent planar flow about a circular cylinder. The simplicity of a two-dimensional flow allows sufficient generality of the unsteady flow conditions while eliminating unnecessary complexities at this stage of development.

Historical Development

Brief History

The current analytical view of the wave force problem for cylinders evolved from the early work of Stokes1 who was concerned with the very slow motion of a pendulum. He found the force to be represented by an expression containing at least two terms. One was proportional to the product of the mass of the fluid displaced by the pendulum and the acceleration. This term is referred to as the inertial term. A second term was proportional to the product of the surface area of the pendulum and its velocity. This linear drag term is characteristic of low Reynolds number or Stokes flow.

In subsequent work Rayleigh2 conjectured that the drag term should scale as the square of the velocity so that it would conform to the empirical form used to describe steady flow. However, he noted that at that time a 3/2 dependency had been proposed by aerodynamicists. Morison3 advanced a two term expression, which is similar to that of Rayleigh and which has been widely used. The Morison equation was advanced on intuitive arguments and empirical evidence. Numerous experimental investigations have been conducted over the past quarter century to determine the empirical inertial and drag coefficients for the Morison equation. All investigations have been characterized by considerable scatter in correlations of the coefficient data.

Wave Force Problem

The general problem of the analysis or design of a marine structure can be divided into at least three aspects: 1) the characterization of the environment (far-field kinematics), 2) the interaction of the structure with the environment (near-field kinematics and wave force methodology), and 3) the optimization of the structure to achieve the maximum benefit at the least risk. Considerable effort has been expended on the environmental aspect in an attempt to characterize the sea state (wave height, current velocity, water particle kinematics, etc.)
to which a structure might be subjected. The description of
the interaction of the sea with an offshore structure through
the wave force methodology predicts the forced and motions
of the structure as a function of the sea-state variables. The
optimization aspect has received considerable attention; how­
ever, progress has suffered from the inadequacy of the
environmental characterization and of the wave force
methodology. The ultimate goal of an optimum structure for
a specified use, life, and location is still in the future.

The difficulties of the environmental aspect have long been
recognized as a source of significant uncertainty in design
problems. Consequently, this aspect has received deserved
attention. Unfortunately, the wave force methodology aspect
of the design of offshore structures has not been as astutely
appraised. Current practice in the design and analysis of
offshore structures and in the analysis of data from scale
experiments assumes the validity of the Morison equation.
The Morison equation is the currently used algorithm for
wave force calculations despite several anomalies. First, a
significant scatter in drag and inertial coefficient data has
persisted through the decades despite the refinement of ex­
perimental techniques. The second anomaly is the ambiguity
in using the Morison equation for a realistic sea state and on a
skewed cylinder where either a two- or three-dimensional
representation is appropriate. Finally, the Morison equation
has proven to be inadequate in correctly accounting for loads
on complex structures.

It has been that, with improved wave kinematic data to
determine the environment either statistically or determi­
nistically and with careful experiments to determine the loads,
the Morison equation could be "forced to fit" conditions of
interest in design. The improvements in load and envi­
ronmental data have not been removed or reduced the
anomalous behavior. The Morison equation has come to be
regarded as an inadequate wave force algorithm for descri­
ing wave-structure interaction with sufficient accuracy to at­
tain the confidence level in design and optimization which po­
tentially should be achieved with existing data.

Development of a Wave Force Methodology

A new wave force methodology capable of surmounting the
current inadequacies can be achieved through several phases
of development. The first or "conceptual phase" involves the
exploration of the physical phenomena present in the problem
and the review of technologies that can be adapted and trans­
ferred to the problem. A fresh and broad view of the problem
is important in this phase. Without the concept no further
development is possible.

To be fully implemented as a methodology, any concept
must pass through phases of development which we term the
"formulation phase," the "validation phase" and, finally,
the interminable "correlation phase." The formulation phase
involves the derivation of analytical expressions or algorithms
for a given physical configuration from the concept. valida­
tion consists of using the formulated expressions with experi­
mental data to indicate that the predictions of the methodol­
y are both plausible and consistent. The correlation phase
involves the processing of large quantities of experimental
data with the methodology to establish relationships between
the far-field kinematics and the near-field kinematics.

An initial investigation of a unified wave force methodol­
gy was undertaken by the author in 19771 as an initial con­
ceptual phase study. One of the concepts from the study is the
vorticity transport integral concept which affords a rational
approach for formulating an analytical wave force methodol­
gy. This paper is intended as a conceptual phase study only.

Vorticity Transport Integral Method

Near-Field Phenomena

The near-field phenomena can be considered to consist of
three parts: 1) diffraction, 2) boundary-layer flow, and
3) wake kinematics.

Diffraction is that phenomenon by which energy is trans­
mitted laterally along a wave crest when a portion of a wave
train is interrupted by a barrier. The convention is to con­
template diffraction to be dominant when the body size is comparable
to the wavelength. However, diffraction is always present as
the process by which the unsteady flow in a wave ac­
commodates to the presence of a body. In short, diffraction
determines the inviscid unsteady flowfield around a structure.

The boundary layer and wake phenomena due to the flow
around a body are closely coupled. Normally, viscous effects
are confined to a thin layer on a body surface, and the fluid
in this momentum deficient layer flows into a thin wake sheat
of the body. This symmetric flow pattern, which is character­
sic of a low Reynolds number flow, is dominated by viscous
drag. As the Reynolds parameter of the flow increases, the
thickness of the boundary layer decreases. When an adverse
pressure gradient (due to deceleration in the flowfield) is
present, the thickness of the boundary layer tends to increase
rapidly as its forward momentum is opposed by both shear
and pressure forces. At some point on the body surface, the
separation point, the retarding shear and pressure forces
cause the flow to break away from the surface and create a re­
circulating wake. This recirculating region will alter the
pressures distribution on both the upstream and downstream
portions of the body. The resulting asymmetry causes the
familiar drag forces associated with high Reynolds number
flows. The separation point is a function of the momentum
transported by the boundary layer. The complexity of trans­
from laminar to turbulent boundary-layer flow also must be
considered.

Direct vs Indirect Approach

The forces on a stationary surface of a body that is wholly
or partially immersed in an unsteady flow are either normal to
the surface and proportional to the pressure or tangential to
the surface and proportional to the shear. Therefore, to
calculate the total force on a body one may simply compute
these two force components on incremental surfaces of the
body and sum these forces vectorially over all surface incre­
ments. This may be termed a direct approach to computing
forces. The direct approach requires an accurate description
of the flowfield especially near the body where gradients in
the flow properties determine the shear and pressure.

Most frequently one takes an indirect approach to com­
puting forces due to flow around a body. In general, the indi­
crt approach consists of integrating the equations of motion
over the flowfield. The resulting expressions will contain in­
egraU of the surface forces that may be replaced by the total
force vectors. Such expressions can be rearranged so that the
total force is expressed as the sum of integrals over the flow­
field. Each of the integrals represents the change of momen­
tum in some portion of the flowfield. Through the device of D'Alembert's principle the integral terms may be
thought of as components of the total force, which are fre­
quently called inertial forces, drag forces, wave-making
forces, etc.

The most widely used indirect approach for the determina­
tion of the total force on a body is the control volume for­
ulation. Usually, certain of the volume integrals are trans­
formed through Green's theorem into surface integrals. The
control volume approach is quite expeditious for steady flows
in that only the surface integrals must be evaluated. For un­
steady flows the usual control volume approach becomes
somewhat clumsy and can be easily misinterpreted. However,
the indirect approach has a significant advantage in that good
accuracy for the integrals can be achieved with an approxi­
mate description of the flowfield. For this reason we have
proposed an alternate indirect approach which we have
terted the Vorticity Transport Integral concept.

Basis of the Concept

The basis for this concept is the vector form of the Navier­
Stokes equations for an incompressible viscous fluid. To sim­
plify the development the usual body forces have been deleted and only a planar flow is discussed (see Fig. 1). These terms are unnecessary to an initial understanding of the concept, and their inclusion leads to undue complexity at this stage. The usual acceleration operator is replaced by Lagrange's relationship

$$\frac{\partial u}{\partial t} + \omega \times u + \text{grad}(q^2/2)$$  \hspace{1cm} (1)$$

where $\omega$ is the vorticity vector and $q^2 = u \cdot u$ the square of the magnitude of the velocity vector. Using this relationship the Navier-Stokes equations can be written as

$$\frac{\partial u}{\partial t} + \omega \times u = - \nabla \vec{p} - \rho \nabla \times \omega$$  \hspace{1cm} (2)$$

where

$$\vec{p} = p \hat{n} + q^2/2$$

The Navier-Stokes equations are integrated over an arbitrary volume of fluid surrounding the body upon which the forces are to be determined. The volume integral terms involving pressure and shear may be converted conveniently into surface integrals by Green's transformation theorem. These surface force integrals can be evaluated in two parts: One integral over the surface adjacent to the body, and the other over the remainder of the surface encompassing the fluid (see Fig. 2). This is accomplished as follows:

$$\int \int \int \vec{v} \cdot \vec{p} \, dV = \int \int \vec{v} \cdot \nabla \times \omega \, dS$$ \hspace{1cm} (3a)$$

$$\int \int \int \vec{v} \times \omega \, dV = \int \int \vec{v} \cdot \nabla \times \omega \, dS$$ \hspace{1cm} (3b)$$

Then the force $F$ on the body is the sum of the shear and pressure forces acting on the surface of the body adjacent to $S_{in}$, which is

$$F = \int \int \vec{v} \cdot \nabla \times \omega \, dS$$ \hspace{1cm} (4)$$

By this device the surface integrals adjacent to the body yield the forces on the body and the integrated equations of motion become

$$F = \int \int \vec{v} \times \omega \, dV - \int \int \vec{v} \cdot \nabla \times \omega \, dV$$ \hspace{1cm} (5)$$

with $\hat{n}$ the outwardly directed normal to the surface element $dS$. Thus, using this expression we can compute the forces on a body by evaluating surface integrals on an arbitrary surface surrounding the body provided the averages of the local acceleration and $\omega \times u$ are known within the volume. The integral expressions presented are comparable to those of a conventional control volume analysis. Needless to say, the evaluation of these integrals over extensive volumes of fluid are formidable and merit a skepticism as to their usefulness.

A significant simplification of the preceding expression results from an identical integration of the far-field description of the flow (the unsteady irrotational flow pattern that, according to convention, would have existed had the body not been present). An integration over a volume identical to that discussed earlier but including the fluid that would have been displaced by the body results in

$$- \int \int \left( \int \rho \frac{\partial \vec{U}}{\partial t} \, dv + \int \rho \frac{\partial \vec{U}}{\partial t} \, dv \right)$$ \hspace{1cm} (6)$$

where $\vec{U}$ is the far-field velocity, $\rho$ the total volume of fluid excluding the body, and $\vec{V}$, the volume displaced by the body. By considering the arbitrary volume of integration to be extensive enough to encompass all near-field effects, i.e.,

$$\int \int \left( \int \rho \mu \nabla \times \omega \, dV + \int \rho \mu \nabla \times \omega \, dV \right)$$ \hspace{1cm} (7)$$

the combination of Eqs. (5) and (6) yields

$$F = \int \int \left( \int \rho \frac{\partial \vec{U}}{\partial t} \, dv + \int \rho \frac{\partial \vec{U}}{\partial t} \, dv \right)$$ \hspace{1cm} (8)$$

The preceding expression has the distinct advantage over both the conventional control volume expression and Eq. (5) in that the integrands have nonzero values only over the near-field portions of the flow. For the flow outside the near-field region $\vec{u} = \vec{U}$ and $\vec{w} = 0$.

The foregoing is intended as a conceptual phase development and not as a detailed formulation phase exposition. The formulation and application studies for various flow situations will be the subject of future efforts.

**Significance of the Integral Terms**

A discussion of the above expressions should begin by recalling that the three integrals in Eq. (7) are all inertial terms that arise from the integration of the acceleration vector over the flowfield. Also, one should note that an explicit drag term is absent.

At first glance, one is tempted to consider the first integral to be the classical inertial term (acceleration evaluated at the centroid of the body times the mass of the fluid displaced by the body) and the second integral to be the virtual mass contribution to the inertial force. It is the third integral that contains the drag contribution. As with many first judgments this appraisal does not stand the test of critical scrutiny.

Let us examine the first integral which is the average of the far-field acceleration over the volume of the body multiplied by the mass of the displaced fluid. Only in the limit of a small body or a negligible spatial variation of the acceleration in the
far field is it equivalent to the acceleration evaluated at the centroid of the body. Therefore, the force contribution from this integral is not necessarily in phase with the inertial force component characterized by the Morison equation. It should also be apparent that, since the force may not act through the centroid of the body, moments about the centroid might also exist due to this term.

Next, consider the third integral which contains the vorticity and was argued to represent the drag contribution. One could argue that the vorticity is associated with the wake and the viscous terms of the boundary-layer flow. These two aspects of the flow are always associated with the concept of the "drag" force. Thus, the third integral must be the kernel of the drag force. To demonstrate that this is not the case is quite simple.

First, consider the clockwise vorticity existing in the wake volume associated with the flow around the upper surface of a body and the counter-clockwise vorticity associated with the flow around the lower surface of the body as depicted in Fig. 3. The cross products of the vectors associated with these two vortices and the convective velocity vectors will produce vectors that are generally perpendicular to the flow (the conventional direction of the drag force is parallel to the flow). Furthermore, if the flow is symmetrical, then these clockwise and counterclockwise contributions will sum to zero. Exceptions to this behavior would exist when the vortices are moving askew to the direction of flow (e.g., during the sweeping of the wake back over the body during flow reversals or when the far-field flow is two-dimensional). However, in the case of wake sweeping this effect would be quite transitory.

The contribution to the third integral arising from the boundary-layer region can be resolved into two force components using the components of the velocity in the boundary layer that are tangential and perpendicular to the surface of the body. The dominant component which is proportional to the square of the local velocity arises from the cross product of the vorticity and the tangential component of the velocity vector. This force component is perpendicular to the front surface of the body and would be directed opposite to the usual direction of the drag force (Fig. 4). The second force component which is $N_s u'$ times the first component arises from the cross product of the vorticity and the normal component of the velocity vector. This results in a force component that is tangent to the body surface but is opposite in direction to the usual drag force. One can demonstrate that there are higher-order terms arising from the vorticity so that this integral will consist of a minimum of four terms which scale as $U'^4$, $U$, $U'^3$, and $U^2$. The coefficients of these terms will be strongly dependent upon the symmetry of the vorticity field.

In summary, the third integral when evaluated over the wake gives rise to lift forces when the flow is asymmetric. When evaluated over the boundary layer this integral also results in negative drag forces. It is quite apparent that the usual drag contribution does not result from this term.

Clearly, the positive or balancing portion of the drag force must arise from the second integral in Eq. (7). To explore the diverse properties of this integral, the volume over which it is evaluated will be expressed as the sum of integrals over the wake volume, the boundary-layer volume, the volume influenced by diffraction and reflection, those volumes which represent previous wake vortices, and the remaining volume of unaffected flow.

Considering the wake volume one finds two contributions. The first is the obvious integral of the rate of change of velocity defect $(U - u)$ over the wake volume frozen in time. The second is due to the expansion or contraction of the wake volume with time. Considering steady flow with this formulation, one finds the wake expansion contribution to be the only contribution from the second integral. This contribution represents the "positive" portion of the drag force which opposes the negative contribution from the vorticity integral.

For an unsteady flow the expansion of the wake volume is far more complex than for steady flow and will vary with the far-field history. With this realization one easily "sees" the problem with the definition of "drag force" in an unsteady flow.

The preceding discussion illustrates some of the qualitative results that can be gleaned from the vorticity transport integral concept without reference to a specific description of the flow. An illustrative example on the quantitative level is quite desirable at this point but is beyond the limits imposed by the intent and the space allocations for this paper. Thus, exploration of integral expressions for a meaningful description of the unsteady flow that is representative of conditions producing wave forces on a submerged body must be deferred to a later paper. However, assurance that these recondite expressions are capable of yielding conventional results can be established through the simplest example of an unsteady flow—the impulsive start of a viscous flow about a circular cylinder.

Illustrative Example

As the simplest illustration of the application of the VTIntegral concept, consider the growth of the wake volume after the impulsive start of flow past a cylinder and the resulting force in the direction of flow. The example will be limited to the behavior of the flow after a steady-state condition has been reached for the far-field flow. For this condition the first integral of Eq. (7) is zero. The third integral contribution has been discussed in the previous section: integration over the symmetric nonspreading wake yields no force contribution in the direction of flow and integration over the forebody boundary.
layer yields a negative drag force which, though important, will not be discussed further.

The object of our attention is the second integral in Eq. (7) and our objective is to illustrate how the positive drag term arises from this integral. The simple model for describing the flow for this illustration consists of the rotational flow about the cylinder, a thin boundary layer on the forebody, and an elongating wake volume extending from the rear surface of the cylinder. Since the control volume $V$ is fixed in time the second integral in Eq. (7) can be written as

$$ F_1 = \int_V \frac{3}{2} \rho \frac{d}{dt} \left( U - u \right) \, dv = \int_V \frac{3}{2} \rho \left( U - u \right) \, dv $$

The control volume can be considered to consist of two volumes (Fig. 5): 1) the volume of the wake and 2) the volume outside the wake region. Although there is a boundary-layer volume also present, the temporal derivative of the average velocity defect over this limited region will be neglected. The change in value of the integral between times $t_1$ and $t_2$ is due to an extension of the wake volume by an amount ($b \Delta t$). Thus the value of the change of the integral over the entire volume in Eq. (8) is given by the changes that are associated with the volume ($b \Delta t$) and represents the difference in the average velocity defect that exists in this region of wake growth from time $t_1$ to time $t_2$. Evaluating Eq. (8) for this case, the component of force in the $x$ direction is

$$ F_{0x} = \frac{L}{2} \int^t_0 \left( \frac{\partial u_x}{\partial t} - \frac{\partial u_x}{\partial t} \right) \, dt = \frac{b \Delta t}{2} \int^t_0 \frac{\partial u_x}{\partial t} \, dt $$

where $L$ is the length of the cylinder, $D$ is the cylinder diameter, $c$ is the velocity of the wake boundary ($dx/dt$), $b$ is the width of the wake volume, and $u_x$ and $\bar{u}_x$ are the average local velocities in the region of wake growth due to the incompressible fluid-body interaction and wake effects, respectively. The assumption that the wake grows only in the $x$ direction (parallel to the flow) has been used to simplify the example. This assumption is not necessary but is sufficient for a qualitative illustration of the role of the second integral term. The form of Eq. (9) is identical to that of the classical drag force for a high Reynolds number steady flow where the wake volume contribution to $C_D$ is

$$ C_D = \frac{2 \left( \frac{\bar{u}_x}{U} - \frac{u_x}{U} \right)}{b \Delta t} $$

Also, as the downstream end of the wake moves away from the vicinity of the body $u_x - U$ which simplifies Eq. (9) somewhat: the contribution, then, from the second integral term of Eq. (7) represents the "positive" portion of the drag force which opposes the negative contribution from the vorticity integral. For a general unsteady flow the expansion of the wake volume is far more complex and will vary with the far-field history.

With the preceding realization one easily sees the problem with the definition of drag force in an unsteady flow. The problem becomes even more difficult when one considers other "volume contributions," spatial variations in the far-field flow, and three-dimensional flow schemes.

**Importance of the Methodology**

The next question is: If the drag and inertia classification of forces is not meaningful, then what methodology should be used? The Vorticity Transport Integral methodology yields several answers to this question. First, one can use the integrals from which the terms arise to classify the force contributions. The first integral is a far-field inertial contribution, the second integral is an inertial defect contribution, and the third integral yields a vorticity transport contribution. Further classification based upon the physical phenomena present is also clearly indicated; the second and third integrals can be divided into integrals over volumes containing the diffraction region, the boundary-layer region, the wake region, etc. This would enable an investigator to determine the significance of each of the volume contributions to the total force for a specific flow scheme.

Clearly, the real power of the vorticity transport integral methodology will be the capacity to supply the level of either quantitative or qualitative understanding which an investigator desires. By using detailed classifications such as that just outlined, one can map out parametrically the range of dominant terms. Because this is an indirect or integral approach, accurate results can be achieved with a modest effort in representing the details of the near- and far-field kinematics. The concept can easily be extended so that differential contributions to the force on a body can be computed. Also, the relative motion of the surface of a system may be considered. Virtually any marine application could be treated when this concept is fully implemented as a methodology.

**Conclusions**

The purpose of this study has been to explore the plausibility of developing a unified wave force theory. A unified theory can be defined as an approach that can predict forces with a desired accuracy over the full range of flow parameters. Such a theory should include a rational representation of the wave kinematics containing dispersive and diffractive factors to the degree that they are an important part of any real wave-structure interaction. The theory must also present an adequate representation of transient viscous flow, considering wake sweeping, asymmetry in the separation pattern of the flow, and three-dimensional effects.

The Vorticity Transport Integral (VTI) concept has been explored and found to afford a meaningful approach to formulating a unified analytical wave force methodology. The VTI method is a rational indirect analytical approach to calculating forces on the surface of a structure and is capable of taking into consideration as many details of a transient flow as an investigator is willing to describe. It is an indirect approach in that it consists of integrating the equations of motion over a flowfield that encompasses all near-field effects. If the volume of the flowfield is separated into isolated near-field regions (boundary layer, wake, etc.), one can evaluate the contribution of each to the total force. Thus, through the use of simple flow models, a comprehensive understanding of the contributions of the near- and far-field phenomena to the force can be achieved.

**References**